OFFPRINT

Controllability of fractal networks: An analytical approach

Jingwen Li, Zhengzhong Yuan, Ying Fan, Wen-Xu Wang
and Zengru Di

Please visit the new website
www.epljournal.org

Controllability of fractal networks: An analytical approach

JINGWEN LI[1], ZHENGZHONG YUAN[1,2(a)], YING FAN, WEN-XU WANG[1] and ZENGRU DI[1]

[1] *School of Systems Science, Beijing Normal University - Beijing, 10085, PRC*
[2] *School of Mathematics and Statistics, Minnan Normal University - Fujian, 363000, PRC*

Abstract – Fractal networks are ubiquitous in nature, ranging from river networks to vascular networks. The ultimate goal of exploring these fractal networked systems lies in controlling the dynamical processes that take place on them. We offer analytical results to exactly understand our ability to control the dynamics of regular fractal networks in terms of identifying the minimum number of driver nodes that are required to achieve full control. According to the exact controllability theory, the controllability of an undirected network is completely determined by the eigenvalue spectrum of the coupling matrix that captures the network structure. The self-similarity in the fractal networks allows us to solve exactly the eigenvalue spectrum from the growth unit and the steps of the iterations, enabling an analytical quantification of the controllability of the fractal networks via the eigenvalue spectrum. We validate our exact analytical results in three typical regular fractal networks. Our results have implications for the control of many real networked systems that have fractal characteristics.

Introduction. – Since it was introduced by Mandelbrot in 1975 [1], the term "fractal" has become the fingerprint of nature. Fractal geometry and patterns have been found to be ubiquitous, such as coastlines, trees, frost crystals, Romanesco broccoli, and more. The development of computer science offers a number of tools for exploring fractal behaviors numerically and mathematically [1,2]. Based on the underlying self-similarity, a variety of iteration models have been proposed to reproduce fractal properties [3–12]. An interesting finding in nonlinear dynamics is that chaotic attractors are often accompanied by fractal structures [13]. In the field of complex networks, fractal properties and self-similarities are shared by many network systems [9–12,14,15], which motivates us to explore how the fractal structure affects the dynamical processes that take place on complex networks. Prototypical approaches include transportation and diffusion [16–19]. Although much effort has been dedicated to exploring the dynamics of fractal networks, how to control the collective dynamics, the ultimate goal of studying them in contemporary science, has not yet been addressed. The recently developed controllability theory of complex networks provides a general framework to understand our ability to

control fractal networks and to achieve full control of the networks [20–25].

In this paper, we explore the controllability of several canonical regular fractal networks that have been constructed from a self-similar iteration process, where the controllability is defined by the fraction of the minimum number of driver nodes that must be controlled to fully steer a networked system [20]. According to the exact controllability theory [25], the controllability of regular fractal networks is solely determined by the eigenvalues of the coupling matrix that characterizes the connections among the nodes. The key thus lies in solving the eigenvalues of the coupling network and linking the eigenvalue spectrum to the controllability, which however is challenging for arbitrary networks but can be exactly accomplished in regular fractal networks. In particular, regular fractal networks with identical weight of edges enable explicit closed-form solutions of the eigenvalue spectrum not only in terms of a thermodynamic limit but also in terms of a finite size. The advantage of regular fractals allows us to offer exact analytical results on the controllability without any approximation. Interestingly, we find that controllability quantified by the fraction of driver nodes can be either an increasing or decreasing function of the network size, depending on the iteration rule of the

(a)E-mail: `zyuan@mnnu.edu.cn`

fractals. Specifically, for a regular fractal network that has completely identical node degrees, it guarantees the absence of degree-degree correlations, and the density of the triangular structures is not altered with the network size; the controllability still decreases remarkably along with the iteration process, which implies that there exists some unknown structural properties (in addition to the node degrees [20] and clustering) that play important roles in the controllability of the complex networks. The controllability of regular fractals of any finite size as well as the evolution of controllability along with the growth of the network can be exactly predicted by our theoretical results. Such an exact theoretical prediction of the controllability of any complex network without an approximation has not been achieved prior to our approach in the literature. Our results have potential applications in controlling real networked systems with fractal structures, such as river networks [26,27] and many biological networks [28–30].

Exact controllability theory. – Consider a controlled network that has N nodes, as described by the following linear ordinary differential equations [20,25]:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \tag{1}$$

where the vector $\mathbf{x} = (x_1, \cdots, x_N)^{\mathrm{T}}$ stands for the states of N nodes, $A \in R^{N \times N}$ denotes the coupling matrix of a complex network, $\mathbf{u}$ is the vector of m controllers: $\mathbf{u} = (u_1, u_2, \cdots, u_m)^{\mathrm{T}}$, and B is the $N \times m$ input matrix. Note that here each node is captured by single state, say, one-dimensional nodal dynamics.

According to ref. [20], the controllability of the complex networked system (1) is determined in terms of finding a matrix B associated with the minimum number of controllers that assure full control of system (1). Such definition is slightly different from that in the classical control theory: system (1) is said to be controllable if there exists a controller $\mathbf{u}$ for a given matrix B that can drive the system state $\mathbf{x}$ from any initial state to any target state. In contrast, here we assume that the matrix B is not fixed and our purpose is to devise a matrix B corresponding to the minimum number of input signals imposed on a minimum set of driver nodes. The controllability of the complex networked system (1) is then defined by the fraction of driver nodes in the sense that a complex network is said to be more controllable if a smaller fraction of driver nodes need to be controlled to achieve full control. In this regard, the minimum number N_{D} of driver nodes is the key to measure the controllability of system (1). According to ref. [25],

$$N_{\mathrm{D}} \equiv \min\{\mathrm{rank}(B)\}. \tag{2}$$

Although one can enumerate all configurations of matrix B by using the Kalman rank condition [31,32] to identify a satisfied B with the minimum rank for low-dimensional A, it is computational prohibitive for large complex networks. Fortunately, it has been proved that [25]

$$N_{\mathrm{D}} = \max_i\{\mu(\lambda_i)\} \tag{3}$$

based on Popov-Belevitch-Hautus rank condition [31,33] that is equivalent to the Kalman rank condition, where λ_i is the eigenvalue of matrix A and the geometric multiplicity $\mu(\lambda_i) \equiv N - \mathrm{rank}(\lambda_i I_N - A)$. It is noted that eq. (3) is valid for any linear coupled system (1).

For a symmetric coupling matrix that has the same geometric multiplicity and algebraic multiplicity [34], such as a matrix of an undirected network, the controllability of the system (1) can be simplified to [25]

$$N_{\mathrm{D}} = \max_i\{\delta(\lambda_i)\}, \tag{4}$$

where $\delta(\lambda_i)$ is the algebraic multiplicity of eigenvalue λ_i and also the eigenvalue degeneracy of matrix A.

For a large sparse network, in which the number of links scales with N in the limit of large N [35], with a small fraction of self-loops, N_{D} is simply determined by the rank of the coupling matrix A [25]:

$$N_{\mathrm{D}} = \max\{1, N - \mathrm{rank}(A)\}, \tag{5}$$

which means the eigenvalue 0 has a maximum multiplicity.

By using eqs. (3), (4) and (5), we can calculate N_{D} of all kinds networks corresponding to the coupling matrix A in eq. (1), including fractal networks. Specifically, for regular fractal networks with undirected edges and identical edge-weight, due to the fact that matrix A is symmetric, we can calculate N_{D} based solely on eq. (4).

According to ref. [20], the controllability of a network is defined by the ratio of N_{D} to the network size N, *i.e.*,

$$n_{\mathrm{D}} = \frac{N_{\mathrm{D}}}{N}. \tag{6}$$

In the following, we analytically derive N_{D} of three types of regular fractal networks with identical edge-weight and consequently their controllability n_{D} with finite and infinite network size.

Modified $(1, 2)$-tree network. – These fractal networks are modified from the famous $(1, 2)$-tree networks [9]. They can be constructed in the following iterative way on every existing edge, as follows: beginning with two adjacent nodes, in each step, replace the edge by a path that is 2 links long, with both endpoints of the path being endpoints of the original edge; then, for each endpoint of the path, create m new nodes and attach them to the endpoint [10]. Figure 1(a) shows a simple illustration of this network. After s steps, we obtain a fractal network T_s with the number of nodes $N_s = (2m+2)^s + 1$ and edges $E_s = (2m + 2)^s$.

It is obvious that the modified $(1, 2)$-tree network is always sparse with $\frac{E_s}{N_s} = \frac{(2m+2)^s}{(2m+2)^s+1} < 1$. Thus, the controllability of this network is totally determined by the degeneracy of the eigenvalue 0. In other words, for the s-step network T_s, the minimum number of driver nodes of T_s is $N_{\mathrm{D}}(T_s) = \max\{1, \delta_s(0)\}$. At this point, we calculate exactly the degeneracy of eigenvalue 0 of the adjacency
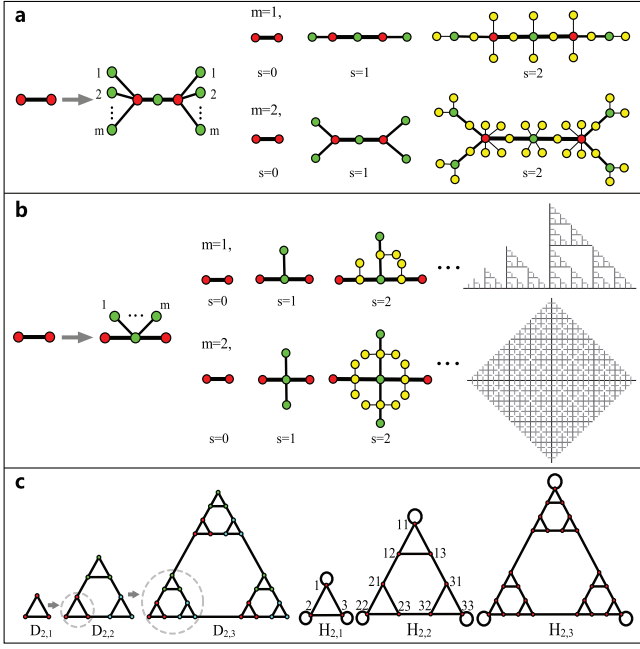
Fig. 1: (Colour on-line) Network model and construction in this paper. (a) General construction of the modified $(1,2)$-tree network and two simple examples of $m = 1$ and $m = 2$ with two steps. (b) General construction of the Peano network and two simple examples of $m = 1$ and $m = 2$ with two and infinite steps. (c) A simple DSGs network and our modified DSGs network for this basic DSGs with added self-loop for each corner node.

matrix, as described in the following: let α represent the set of nodes that belong to the s-step network T_s, and let β be the set of nodes that are generated at the $(s+1)$-th iteration. From the construction, the adjacency matrix of T_{s+1} has the following block form:

$$
\begin{aligned}
A_{s+1} &= \begin{pmatrix} A_{\alpha,\alpha} & A_{\alpha,\beta} \\ A_{\beta,\alpha} & A_{\beta,\beta} \end{pmatrix} \\
&= \begin{pmatrix} 0 & A_{\alpha,\beta} \\ A_{\beta,\alpha} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & A_{\beta,\alpha}^{\mathrm{T}} \\ A_{\beta,\alpha} & 0 \end{pmatrix}.
\end{aligned}
\tag{7}
$$

Then, we can obtain the degeneracy of eigenvalue 0 as

$$
\begin{aligned}
\delta_{s+1}(0) &= N_{s+1} - \mathrm{rank}(A_{s+1}) \\
&= N_{s+1} - \mathrm{rank}(A_{\beta,\alpha}) - \mathrm{rank}(A_{\beta,\alpha}^{\mathrm{T}}) \\
&= N_{s+1} - 2\mathrm{rank}(A_{\beta,\alpha})
\end{aligned}
\tag{8}
$$

where $A_{\beta,\alpha}$ is a matrix with $N_{s+1} - N_s$ rows and N_s columns and is written as follows:

$$
A_{\beta,\alpha} = \begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,N_s} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,N_s} \\
\vdots & \vdots & \ddots & \vdots \\
a_{N_{s+1}-N_s,1} & a_{N_{s+1}-N_s,2} & \cdots & a_{N_{s+1}-N_s,N_s}
\end{pmatrix}.
\tag{9}
$$

It has been proven that the N_s columns of $A_{\beta,\alpha}$ are linearly independent [10], i.e., $\mathrm{rank}(A_{\beta,\alpha}) = N_s = (2m+2)^s + 1$. Thus, $\delta_{s+1}(0) = N_{s+1} - 2N_s = 2m(2m+2)^s - 1$, and

$$
N_{\mathrm{D}}(T_s) = \max\{1, \delta_s(0)\} = \begin{cases} 1, & s = 0, \\ 2m(2m+2)^{s-1} - 1, & s \geq 1. \end{cases}
\tag{10}
$$

The controllability n_{D} of the modified $(1,2)$-tree network is given by

$$
n_{\mathrm{D}} = \frac{N_{\mathrm{D}}(T_s)}{N_s} = \frac{2m(2m+2)^{s-1} - 1}{(2m+2)^s + 1}
\tag{11}
$$

with its thermodynamic limit

$$
\lim_{s \to \infty} n_{\mathrm{D}} = \lim_{s \to \infty} \frac{2m(2m+2)^{s-1} - 1}{(2m+2)^s + 1} = \frac{m}{m+1}
\tag{12}
$$

for any given m.

Peano network. – The Peano network [26,27] can also be constructed in an iterative way, as follows: begin with two adjacent nodes. In each step, insert a new node to each existent edge and for each newly added node, link m new nodes to it. After s steps, we obtain a fractal network P_s, as shown in fig. 1(b). The edges of P_s and P_{s-1} have the relations $E_s = 2E_{s-1} + mE_{s-1} = (2 + m)E_{s-1}$ with $E_0 = 1$. In parallel, the nodes have similar relations, i.e., $N_s = N_{s-1} + (m+1)E_{s-1} = N_{s-1} + (m+1)(m+2)^{s-1}$ with $N_0 = 2$. In other words, after s steps, we obtain the Peano network P_s, with the number of nodes $N_s = (m+2)^s + 1$ and edges $E_s = (m+2)^s$.

Because this Peano network is also sparse with $\frac{E_s}{N_s} = \frac{(m+2)^s}{(m+2)^s+1} < 1$, its controllability is then totally determined by the degeneracy of eigenvalue 0, in other words, $N_{\mathrm{D}}(P_s) = \max\{1, \delta_s(0)\}$. Thus, similar to the modified $(1,2)$-tree network, we consider only the degeneracy of the eigenvalue 0 of the adjacency matrix.

Analogous to the modified $(1,2)$-tree network, let α represent the set of nodes that belong to the s-step Peano network P_s with N_s nodes, and let β be the set of newly added nodes that were generated at the $(s + 1)$-th iteration with $\beta = \beta_1 + \beta_2$, where β_1 represents the set of new nodes inserted into the edges with $N_s^{(\beta_1)} = E_s = N_s - 1$, and β_2 represents the set of new nodes that are adjacent to the inserted nodes with $N_s^{(\beta_2)} = mN_s^{(\beta_1)} = m(N_s - 1)$. From the construction, the adjacency matrix of P_{s+1} has the following block form:

$$
\begin{aligned}
A_{s+1} &= \begin{pmatrix} A_{\alpha,\alpha} & A_{\alpha,\beta_2} & A_{\alpha,\beta_1} \\ A_{\beta_2,\alpha} & A_{\beta_2,\beta_2} & A_{\beta_2,\beta_1} \\ A_{\beta_1,\alpha} & A_{\beta_1,\beta_2} & A_{\beta_1,\beta_1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & A_{\alpha,\beta_1} \\ 0 & 0 & A_{\beta_2,\beta_1} \\ A_{\beta_1,\alpha} & A_{\beta_1,\beta_2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (A_{\beta_1,\alpha}, A_{\beta_1,\beta_2})^{\mathrm{T}} \\ (A_{\beta_1,\alpha}, A_{\beta_1,\beta_2}) & 0 \end{pmatrix}.
\end{aligned}
\tag{13}
$$

$$(A_{\beta_1,\alpha}, A_{\beta_1,\beta_2}) = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N_s} & 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N_s} & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{N_s-1,1} & a_{N_s-1,2} & \cdots & a_{N_s-1,N_s} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \end{pmatrix}. \quad (15)$$

Then, we can calculate the degeneracy of eigenvalue 0 as follows:

$$\begin{aligned} \delta_{s+1}(0) &= N_{s+1} - \text{rank}(A_{s+1}) \\ &= N_{s+1} - \text{rank}(A_{\beta_1,\alpha}, A_{\beta_1,\beta_2}) \\ &\quad - \text{rank}((A_{\beta_1,\alpha}, A_{\beta_1,\beta_2})^{\text{T}}) \\ &= N_{s+1} - 2\text{rank}(A_{\beta_1,\alpha}, A_{\beta_1,\beta_2}), \quad (14) \end{aligned}$$

where $(A_{\beta_1,\alpha}, A_{\beta_1,\beta_2})$ is a matrix with $N_s - 1$ rows and $N_s + m(N_s - 1)$ columns which can be written as

see eq. (15) above

It is obvious that the rows that correspond to the last $m(N_s - 1)$ columns of A_{β_1,β_2} are orthogonal, which means that they are linearly independent [34], *i.e.*, $\text{rank}(A_{\beta_1,\beta_2}) = N_s - 1 = (m + 2)^s$. Following this conclusion, the whole $N_s - 1$ rows are also linearly independent [34]. In other words, $\text{rank}(A_{\beta_1,\alpha} \ A_{\beta_1,\beta_2}) = \text{rank}(A_{\beta_1,\beta_2}) = (m + 2)^s$. Thus, $\delta_{s+1}(0) = m(m + 2)^s + 1$ and

$$N_{\text{D}}(P_s) = \max\{1, \delta_s(0)\} = \begin{cases} 1, & s = 0, \\ m(m + 2)^{s-1} + 1, & s \geq 1. \end{cases} \quad (16)$$

The controllability n_{D} of the Peano network can then be calculated as

$$n_{\text{D}} = \frac{N_{\text{D}}(P_s)}{N_s} = \frac{m(m + 2)^{s-1} + 1}{(m + 2)^s + 1} \quad (17)$$

with its thermodynamic limit

$$\lim_{s \to \infty} n_{\text{D}} = \lim_{s \to \infty} \frac{m(m + 2)^{s-1} + 1}{(m + 2)^s + 1} = \frac{m}{m + 2} \quad (18)$$

for any m.

Modified dual Sierpinski gaskets network. – This fractal network is an extension of the dual Sierpinski gaskets (DSGs), which is in turn an extension of the basic Sierpinski gaskets (SGs). Let $D_{d,s}$ denote d-dimension DSGs after s generations with $d \geq 2$ and $s \geq 0$. $D_{d,0}$ represents a fully connected network with $d + 1$ nodes. $D_{d,s}$ is iterated by $D_{d,s-1}$ as follows: combining $d + 1$ copies of $D_{d,s-1}$ by adding some extra edges that connect corner nodes with the smallest degree d in the copies of $D_{d,s-1}$, for example, as shown in fig. 1(c). This fractal network has $d + 1$ corner nodes. Our modified DSGs are basic DSGs with added self-loops for each $d + 1$ corner node, where each node has the same degree [11].

A simple example of our modified DSGs is also shown in fig. 1(c). We also use $D_{d,s}$ to represent our modified DSGs without confusion. After s steps, we obtain the modified DSGs network $D_{d,s}$ with the number of nodes $N_{d,s} = (d+1)^{s+1}$ and edges $E_{d,s} = \frac{(d+1)^{s+2} + (d+1)}{2}$, which leads to $\frac{E_{d,s}}{N_{d,s}} = \frac{(d+1)^{s+2} + (d+1)}{2(d+1)^{s+1}} \approx \frac{d+1}{2}$; this result means that it is also a sparse network. Thus, the controllability of our modified DSGs network is also totally determined by the degeneracy of the eigenvalue 0, in other words, $N_{\text{D}}(D_{d,s}) = \max\{1, \delta_s(0)\}$. Here, we need to calculate only the multiplicity of the eigenvalue 0 in the following.

For this modified DSG network, we consider the coupled system (1), in which the coupling matrix A is the transition matrix. The off-diagnal element in matrix A is defined as $a_{ij} = \frac{1}{d+1}$ if the nodes i and j are adjacent, otherwise $a_{ij} = 0$; for the diagnal element $a_{ii} = \frac{1}{d+1}$ if the node i is one of the $d + 1$ corner nodes, otherwise $a_{ii} = 0$. Using the decimation technique [12], it has been proven that the degeneracy for the eigenvalue 0 has the following relation [11]:

$$\delta_{s+1}(0) = (d+1)\delta_s(0) + \frac{d(d+1)}{2} \quad (s = 0, 1, 2, \cdots) \quad (19)$$

with $\delta_0(0) = d$ because all of the elements of this network's transition matrix are $\frac{1}{d+1}$. Thus, the multiplicity of eigenvalue 0 is

$$\delta_s(0) = \frac{d-1}{2}(d+1)^s + \frac{d+1}{2} \quad (20)$$

and

$$\begin{aligned} N_{\text{D}}(D_{d,s}) &= \max\{1, \delta_s(0)\} \\ &= \frac{d-1}{2}(d+1)^s + \frac{d+1}{2} \quad (s = 0, 1, 2, \cdots). \end{aligned} \quad (21)$$

The controllability n_{D} of our modified DSGs coupled with the transition matrix becomes

$$n_{\text{D}} = \frac{N_{\text{D}}(D_{d,s})}{N_s} = \frac{\frac{d-1}{2}(d+1)^s + \frac{d+1}{2}}{(d+1)^{s+1}} \quad (22)$$

with its thermodynamic limit

$$\lim_{s \to \infty} n_{\text{D}} = \lim_{s \to \infty} \frac{\frac{d-1}{2}(d+1)^s + \frac{d+1}{2}}{(d+1)^{s+1}} = \frac{d-1}{2(d+1)} \quad (23)$$
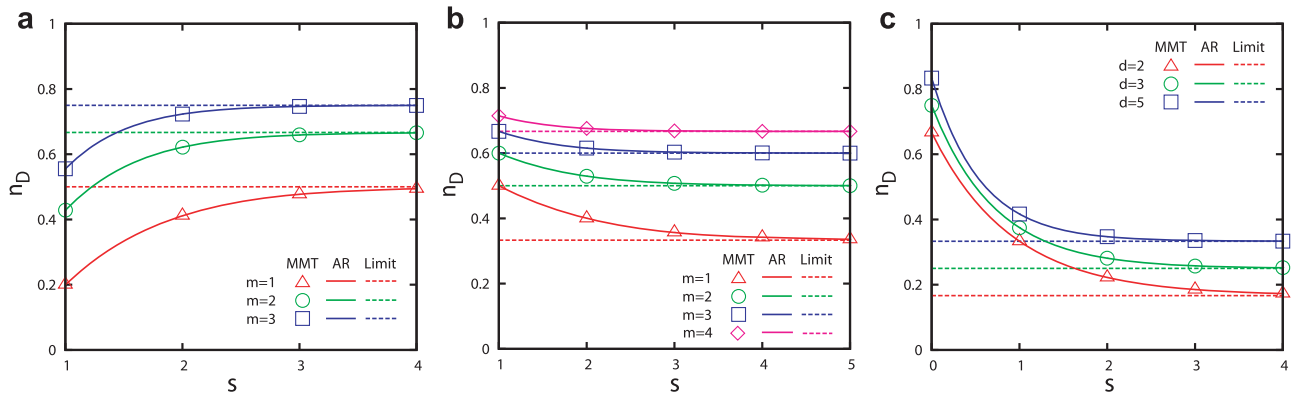
for any given m.

Fig. 2: (Colour on-line) Controllability measure n_D as a function of iteration step s for (a) modified $(1, 2)$-tree networks, (b) Peano networks and (c) modified DSGs networks, respectively. MMT denotes the results by numerically calculating eq. (4), AR denotes the analytical results predicted by eqs. (11), (17) and (22) for the three types of networks, respectively, and Limit denotes the thermodynamic limit predicted by eqs. (12), (18) and (23) for the three networks, respectively.

Numerical test and discussion. – We numerically test the analytical results of the controllability n_D of $(1, 2)$-tree Peano and modified DSGs networks, respectively. As shown in fig. 2, we see that the analytical results obtained from eqs. (11), (17) and (22), respectively, for the three types of fractal networks are in exactly agreement with numerical calculations based on the MMT theory (eq. (4)). For the minimum number of driver nodes N_D, the three types of networks share a common feature: N_D increases exponentially as the iteration step s increases, as reflected in eqs. (10), (16) and (21). In other words, N_D increases very fast with the growth of the network. However, the controllability measure n_D is limited to a constant lower than 1, as predicted by eqs. (12), (18) and (23) for the three networks, respectively. This seemingly counterintuitive finding is due to the fact that the network size is also exponentially increases with the iteration step s, as can be found in the denominator of eqs. (11), (17) and (22) for the three networks, respectively, accounting for the existent of the constant thermodynamic limits and the fast approaching to the limit as s increases, as shown in fig. 2.

Apart from the common characteristic of n_D in the fractal networks, the results of the modified DSGs raise an interesting question that pertains to the structural effect on the network controllability. In the modified DSGs, all of the degrees of the nodes are fixed to be $d + 1$, regardless of the iteration in the self-similarly growth, which also ensures the absence of degree-degree correlation. The density of the triangular structure in the DSGs is invariant as well. However, the controllability n_D decreases dramatically, which suggests that there are some unknown structural properties that play significant roles in the controllability other than node degrees, degree-degree correlation and triangular structure. This conjecture raises the need to design subtle schemes to uncover the impact of each structural property on the controllability by screening the effects of others.

Conclusions. – We have demonstrated that the controllability of fractal networks that consist of a broad class of complex networks can be exactly predicted by employing the exact controllability theory. In particular, for sparse fractal networks, according to the exact controllability theory, the controllability is completely determined by the multiplicity of the zero eigenvalue, which can be exactly derived because of the self-similarity of the network. Three prototypical regular fractal networks have been explored to validate our analytical results, including the modified $(1, 2)$-tree network, the Peano network and the modified dual Sierpinski gaskets. Our analytical results are in exact agreement with those of immediately using the exact controllability theory to compute the maximum algebraic multiplicity of the coupling matrix. Our analytical results are valid not only for finite network sizes but also for the thermodynamic limit. Although we focus on the controllability of fractal networks, our work offers a general paradigm to explore the controllability of general networks by bridging the eigenvalue spectrum and controllability theory, opening new avenues to achieve the control of real systems.

∗ ∗ ∗

REFERENCES

[1] MANDELBROT B., *The Fractal Geometry of Nature* (Freeman, San Francisco, Cal.) 1982.
[2] FALCONER K. J., *Fractal Geometry: Mathematical Foundations and Applications* (Wiley, Chichester) 2003.

[3] SIERPINSKI W., *C. R. Acad. Sci. Paris*, **160** (1915) 302.

[4] HATTORI T. and TSUDA T., *J. Stat. Phys.*, **109** (2002) 39.

[5] VON KOCH H., *Acta Math.*, **30** (1906) 145.

[6] LAKHTAKIA A., VARADAN V. K., MESSIER R. and VARADAN V. V., *J. Phys. A: Math. Gen.*, **20** (1987) 3537.

[7] VICSEK T., *J. Phys. A: Math. Gen.*, **16** (1983) L647.

[8] ZHANG Z. Z., WU B., ZHANG H. J., ZHOU S. G., GUAN J. H. and WANG Z. G., *Phys. Rev. E*, **81** (2010) 031118.

[9] ROZENFELD H. D., HAVLIN S. and BEN-AVRAHAM D., *New J. Phys.*, **9** (2007) 175.

[10] ZHANG Z., HU Z., SHENG Y. and CHEN G., *EPL*, **99** (2012) 10007.

[11] WU S. and ZHANG Z., *J. Phys. A: Math. Theor.*, **45** (2012) 345101.

[12] KLAVŽAR S. and MILUTINOVIĆ U., *Czech. Math. J.*, **47** (1997) 95.

[13] AGUIRRE J., VIANA R. L. and SANJUÁN M. A. F., *Rev. Mod. Phys.*, **81** (2009) 333.

[14] SONG C., HAVLIN S. and MAKSE H. A., *Nature*, **433** (2005) 392.

[15] SONG C., HAVLIN S. and MAKSE H. A., *Nat. Phys.*, **2** (2006) 275.

[16] KATIFORI E., SZÖLLŐSI G. J. and MAGNASCO M. O., *Phys. Rev. Lett.*, **104** (2010) 048704.

[17] HU D. and CAI D., *Phys. Rev. Lett.*, **111** (2013) 138701.

[18] CORSON F., *Phys. Rev. Lett.*, **104** (2010) 048703.

[19] DIMITROV P. and ZUCKER S. W., *Proc. Natl Acad. Sci. U.S.A.*, **103** (2006) 9363.

[20] LIU Y.-Y., SLOTINE J.-J. and BARABÁSI A.-L., *Nature*, **473** (2011) 167.

[21] NEPUSZ T. and VICSEK T., *Nat. Phys*, **8** (2012) 568.

[22] YAN G., REN J., LAI Y.-C., LAI C.-H. and LI B., *Phys. Rev. Lett.*, **108** (2012) 218703.

[23] LIU Y.-Y., SLOTINE J.-J. and BARABÁSI A. L., *PLoS ONE*, **7** (2012) e44459.

[24] WANG W.-X., NI X., LAI Y.-C. and GREBOGI C., *Phys. Rev. E*, **85** (2012) 026115.

[25] YUAN Z., ZHAO C., DI Z., WANG W.-X. and LAI Y.-C., *Nat. Commun.*, **4** (2013) 2447.

[26] BERTUZZO E., MARITAN A., GATTO M., RODRIGUEZ-ITURBE I. and RINALDO A., *Water Resour. Res.*, **43** (2007) W04419.

[27] GATTO M., MARI L., BERTUZZO E., CASAGRANDI R., RIGHETTO L., RODRIGUEZ-ITURBE I. and RINALDO A., *Proc. Natl Acad. Sci. U.S.A.*, **109** (2012) 19703.

[28] WEST G., BROWN J. H. and ENQUIST B. J., *Science*, **276** (1997) 122.

[29] BANAVAR J., MARITAN A. and RINALDO A., *Nature*, **399** (1999) 130.

[30] GARLASCHELLI D., CALDARELLI G. and PIETRONERO L., *Nature*, **423** (2003) 165.

[31] RUGH W. J., *Linear System Theory*, 2nd edition (Prentice-Hall, NJ) 1996.

[32] KALMAN R. E., *J. Soc. Ind. Appl. Math. Ser. A*, **1** (1963) 152.

[33] HAUTUS M. L. J., *Ned. Akad. Wet., Proc. Ser. A*, **72** (1969) 443.

[34] STRANG G., *Linear Algebra and Its Applications*, 4th edition (Thomson Brooks/Cole, Belmont, Cal.) 2006.

[35] NEWMAN M. E. J., *Phys. Rev. E*, **69** (2004) 066133.